Stability of Faired Underwater Towing Cables

S. Nair*
Illinois Institute of Technology, Chicago, Ill.
and
G. Hegemier†
University of California, San Diego, Calif.

The dynamic stability of faired, heterogeneous underwater cables subject to small perturbations from a planar towing configuration is considered. Sufficient conditions for freedom from flutter and divergence are obtained using the velocity component strip theory. These conditions are given in the form of constraints on the locations of shear center, center of tension, hydrodynamic center, and center of gravity of the cable cross section.

I. Introduction

IN a previous paper[1] a three-dimensional dynamic structural model suitable for the treatment of heterogeneous, transverse isotropic cable systems was developed. The theory presented in Ref. 1 admits large amplitude motions under the constraints of small strains and includes bending and torsional effects as well as the usual extensional behavior. In what follows, the theory developed in Ref. 1 is employed to study the steady-state behavior as well as the modes of instability such as divergence and flutter of heterogeneous, transverse isotropic underwater cables in subcavitating environments.

With the advent of high-speed marine vehicles such as the hydrofoil, problems associated with cable-towing deeply submerged sensors at high speeds from surface crafts have received increasing attention. A major limitation on the speed and depth attainable in each case is the power required to tow the system. The latter is largely a function of the drag characteristics of the cable. Thus, because of excessive drag, conventional cables with circular or stranded cross section, or similar cables with clip-on fairings, have been found to be unsatisfactory for high-speed applications.

In an effort to minimize drag forces, recent designs have focused upon continuously faired cables, i.e., cables with a streamlined, airfoil-shaped cross section, continuous along the span. An example of such a cable is that developed by the Boeing Company (Fig. 1). This design utilizes a NACA 63A022 section. The strength member is molded fiber glass. A flexible rubber trailing edge is bonded to the strength member to provide the necessary aft contour of the airfoil.

As with a wing or hydrofoil, streamlined cross sections may lead to hydroelastic instabilities in the form of divergence or flutter. Neglecting to account for such phenomena may, in practice, result in guidance problems, unnecessarily large noise levels, increased power requirements, and/or structural failure.

In contrast to the current and past literature on the traditional wing and hydrofoil problems, only a few incomplete works presently exist on the subject of faired-tow cable stability.[2,3] Even in the case of round cables the structural modeling often neglects the bending stiffness of the cable[3] or the elasticity of the cable altogether.[4]

Received March 16, 1978; revision received July 14, 1978.

Index category: Marine Mooring Systems and Cable Mechanics.
*Visiting Assistant Professor, Dept. of Mechanics and Mechanical and Aerospace Engineering. Member AIAA.
†Professor of Applied Mechanics and Engineering Sciences. Member AIAA.

The planar towing configurations of cables have received considerable attention in the past.[5-7] In Sec. II we briefly discuss the problem of planar steady-state towing. Differential equations governing the local stability of planar towing configurations are presented in Sec. III. In the remaining sections sufficient conditions for freedom from divergence and flutter are obtained in the case of towing as well as in the case of tethering cables.

II. Planar Steady-State Towing

We consider a uniform cable of length l with cross sections symmetric about the plane of towing (Fig. 2). Selecting the locus of material points along the cross-sectional shear centers as the reference curve $\mathcal{C}$ we have from Ref. 1 for $0<s<l$,

$$dN/ds+\kappa T+f_1^{(H)}+f_1^{(G)}=0 \tag{1a}$$

$$dT/ds-\kappa N+f_3^{(H)}+f_3^{(G)}=0 \tag{1b}$$

$$dM/ds+N=0 \tag{1c}$$

$$T=\overline{EA}(e-\theta_n\kappa) \tag{2a}$$

$$M=\overline{EI_{11}}\kappa-\overline{EA}\theta_n e \tag{2b}$$

where T, M, κ, and e represent shearing force, cable tension, bending moment, curvature, and extensional strain of the reference curve, respectively, and where $f_i^{(H)}$, $f_i^{(G)}$ represent components of hydrodynamic and gravity forces on the cable with respect to an orthogonal unit triad A_i fixed on the reference curve at s, as shown in Fig. 3. With coordinates θ_1, θ_2, θ_3 $(=s)$ taken along the base vectors A_1, A_2, A_3, the quantities $\overline{EA}$, $\overline{EI_{\alpha\beta}}$ are defined as

$$\overline{EA}=\iint E(\theta_1,\theta_2)\,d\theta_1 d\theta_2$$

$$\overline{EI_{\alpha\beta}}=\iint E(\theta_1,\theta_2)\theta_\alpha\theta_\beta\,d\theta_1 d\theta_2 \tag{3}$$

where E is Young's modulus and α, β take value 1 or 2. In Eq. (2) θ_n represents the distance between the shear center and the center of tension (neutral axis) of the cable cross section, with θ_n being algebraically positive if the center of tension lies in between the shear center and the leading edge.

We assume that the cable is attached to the towing vehicle and the towed body through ball or hinge joints located at the center of tension. Then, the boundary conditions are

$$M(0)=-T(0)\theta_n,\; M(l)=-T(l)\theta_n \tag{4a}$$

$$N(l)=L\sin\varphi(l)-D\cos\varphi(l) \tag{4b}$$

$$T(l)=L\cos\varphi(l)+D\sin\varphi(l) \tag{4c}$$

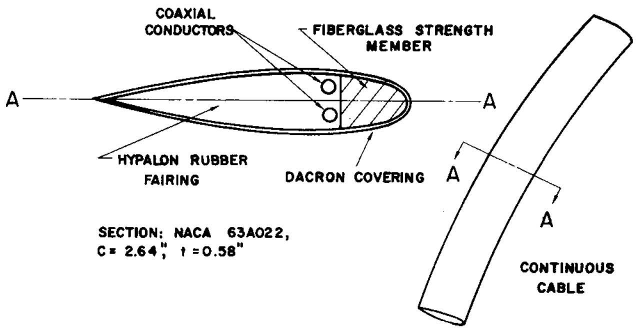

Fig. 1 Details of the Boeing cable.

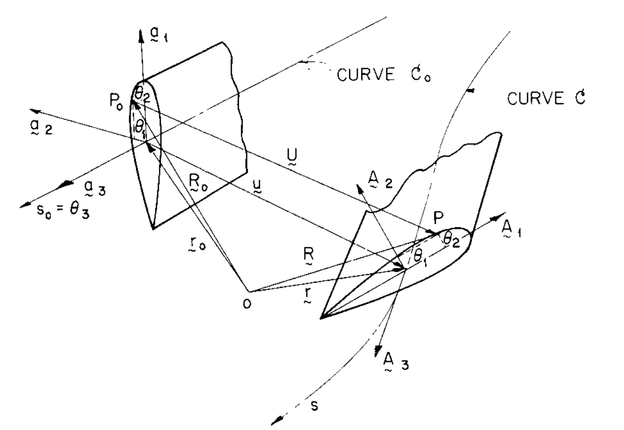

Fig. 2 Cable coordinate systems; $\mathcal{C}_0$ represents the elastic axis of the underformed cable and $\mathcal{C}$ represents that of the deformed cable in steady state.

where L, D denote downward lift and drag forces on the lower cable terminus by the towed body and φ is the angle between the vertical and the local tangent to $\mathcal{C}$.

Eliminating M and N, we write the system of Eqs. (1) and (2) in the form

$$(\overline{EI}_{11} - \overline{EA}\theta_n^2)\,d^2\kappa/ds^2 - \theta_n d^2T/ds^2 - \kappa T - f_1^{(H)} - f_1^{(G)} = 0 \tag{5}$$

$$(1-\theta_n\kappa)dT/ds + (\overline{EI}_{11} - \overline{EA}\theta_n^2)\kappa\, d\kappa/ds + f_3^{(H)} + f_3^{(G)} = 0 \tag{6}$$

We note that the preceding system of equations could be written in a simpler form if the neutral axis is used as a refererence curve $\mathcal{C}$. However, in the context of a stability analysis we find it more appropriate to use cable variables referred to the locus of shear centers (elastic axis).

In order to assess the orders of magnitude of the various terms in Eqs. (5) and (6) we introduce the following nondimensional quantities,

$$\tilde T = T/\bar T,\ \tilde\kappa = \kappa/\bar\kappa,\ \zeta = s/R \tag{7a}$$

$$\tilde f_l = R(f_l^{(H)} + f_l^{(G)})/\bar T,\ \tilde L = L/\bar T,\ \tilde D = D/\bar T \tag{7b}$$

$$\epsilon = \pi b/l,\ \tilde\theta_n = \theta_n/b,\ \tilde B_l = \pi(\overline{EI}_{11} - \overline{EA}\theta_n^2)/\bar T bl \tag{7c}$$

where $\bar\kappa$, $\bar T$ represent certain average values of κ and T over the span s, and $R \equiv 1/\bar\kappa$ is the characteristic length associated with significant variations of the steady-state variables. In Eqs. (7), b represents half-chord and all quantities $(\tilde{\ })$ are at most $O(1)$. As a criterion for the validity of the cable equations, $\epsilon \ll 1$. In Eqs. (7) we have taken into consideration the fact that the hydrodynamic drag forces acting normal and tangential to the reference curve may be of the same order of magnitude for faired cables.[3]

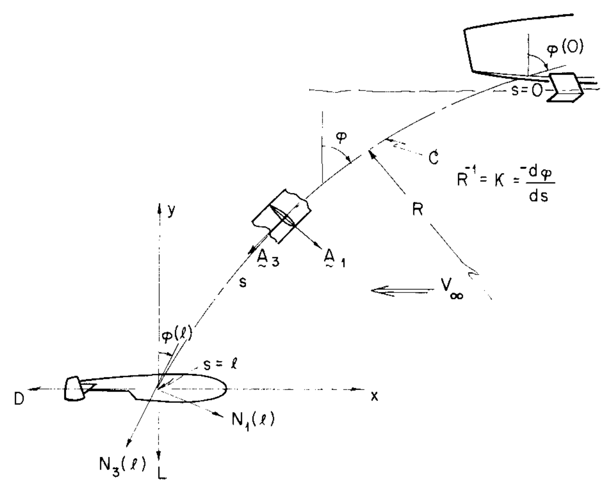

Fig. 3 Steady-state towing configuration.

Substituting Eqs. (7) into Eqs. (5) and (6) we obtain

$$\epsilon\left(\frac{l}{\pi R}\right)^2\left[\tilde B_l\frac{\partial^2\tilde\kappa}{\partial\zeta^2} - \tilde\theta_n\frac{\partial^2\tilde T}{\partial\zeta^2}\right] - \tilde\kappa\tilde T - \tilde f_l = 0 \tag{8a}$$

$$\left(1-\tilde\theta_n\epsilon\frac{l}{\pi R}\right)\frac{\partial\tilde T}{\partial\zeta} + \epsilon\left(\frac{l}{\pi R}\right)^2\tilde\kappa\frac{\partial\tilde\kappa}{\partial\zeta} + \tilde f_3 = 0 \tag{8b}$$

The boundary conditions of Eqs. (4) may be transformed with the use of Eqs. (7) to read,

$$\epsilon\left(\frac{l}{\pi R}\right)^2\tilde\kappa(0) = \epsilon\left(\frac{l}{\pi R}\right)^2\tilde\kappa\left(\frac{l}{R}\right) = 0,$$

$$\tilde T\left(\frac{l}{R}\right) = \tilde L\cos\varphi(l) + \tilde D\sin\varphi(l) \tag{9}$$

$$\epsilon\frac{l}{\pi R}\left[\tilde\theta_n\frac{d\tilde T}{d\zeta}\left(\frac{l}{R}\right) - \frac{l}{\pi R}\tilde\beta_l\frac{d\tilde\kappa}{ds}\left(\frac{l}{R}\right)\right] = \tilde L\sin\varphi(l) - \tilde D\cos\varphi(l) \tag{10}$$

From Eqs. (9) and (10) we conclude that the terms containing ϵ are negligible as long as $l \le O(\pi R)$. Corresponding to this approximation, Eqs. (9) and (10) reduce to the standard cable equations,

$$dT/ds + f_3^{(H)} + f_3^{(G)} = 0,\ T\kappa + f_l^{(H)} + f_l^{(G)} = 0 \tag{11}$$

and the boundary conditions

$$T(l) = L\cos\varphi(l) = D\sin\varphi(l),\ \tan\varphi(l) = D/L \tag{12}$$

We note that the preceding approximate system of equations is valid in the interior of the cable, away from two end zones (boundary layers) of thickness $O(\sqrt{\epsilon l})$. Within these end zones the curvature κ varies from 0 to its value in accordance with Eq. (10). Most cable analyses in the literature are based entirely upon equations of the type (10) whereby bending effects are neglected. Consequently, the solutions of Eq. (10) have been adequately discussed. Details concerning suitable loading functions f_l and f_3 and analytical as well as numerical solution methods may be found in Ref. 3 or in the references cited therein. The purpose of the foregoing analysis is to bring out the approximations involved in the standard steady-state cable equations so that our approximations in what follows would be consistent with these fundamental approximations.

III. Equations for Local Stability of Planar Steady-State Solution

In the context of local stability analysis (initiation of an unstable motion) we consider the dynamic state of the cable composed of the planar steady state and a small disturbance in the form of three displacement components and a rotation which are functions of time and the spatial variable s.

With respect to the elastic axis, the nonzero steady-state cable variables T, N, M, and κ are related in the form

$$N = -\mathrm{d}M/\mathrm{d}s, \quad M = (\overline{EI_{11}} - \overline{EA}\theta_n^2)\kappa - T\theta_n \tag{13}$$

Let us now superpose a disturbance on the steady-state motion according to

$$\eta(s,t) = \hat{\eta}(s) + \eta^*(s,t) \tag{14}$$

where η represents any of the cable variables listed in Ref. 1, $\hat{\eta}$ represents the planar steady variables, and η^* represents a small disturbance. We have $\hat{\eta}(s) = 0$, except when $\hat{\eta}$ represents $\hat{N}_1$, $\hat{N}_3$, $\hat{M}_2$, $\hat{\kappa}_1$, or $\hat{e}$. Consistent with our notation, the nonzero values of $\hat{\eta}$ are

$$\hat{N}_1 = N, \quad \hat{N}_3 = T, \quad \hat{M}_2 = M, \quad \hat{\kappa} = \kappa, \quad \hat{e} = e \tag{15}$$

Upon substituting Eq. (14) into Eqs. (58-62) of Ref. 1 and retaining only linear terms in η^* we obtain

$$N_1^{*\prime} + \kappa N_3^* + T\kappa_1^* + f_1^{(I)*} + f_1^{(H)*} = 0 \tag{16a}$$

$$N_2^{*\prime} + N\kappa_3^* + T\kappa_2^* + f_2^{(I)*} + f_2^{(H)*} = 0 \tag{16b}$$

$$N_3^{*\prime} - \kappa N_1^* - N\kappa_1^* + f_3^{(I)*} + f_3^{(H)*} = 0 \tag{16c}$$

$$M_1^{*\prime} - M\kappa_3^* + \kappa M_3^* - N_2^* = 0 \tag{17a}$$

$$M_2^{*\prime} + N_1^* = 0 \tag{17b}$$

$$M_3^{*\prime} - \kappa M_1^* - M_2\kappa_2^* + m_3^{(I)*} + m_3^{(H)*} = 0 \tag{17c}$$

$$N_3^* = \overline{EA}(e^* - \theta_n\kappa_1^*) \tag{18a}$$

$$M_1^* = -\overline{EI_{22}}\kappa_2^* \tag{18b}$$

$$M_2^* = (\overline{EI_{11}} - \overline{EA}\theta_n^2)\kappa_1^* - N_3^*\theta_n \tag{18c}$$

$$M_3^* = \overline{GJ_{ef}}\kappa_3^* \tag{18d}$$

where primes denote differentiation with respect to s and where $\overline{EI_{22}}$, $\overline{EI_{11}}$ are the bending rigidities around the A_1, A_2 axes, $\overline{GJ_{ef}}$ the torsional rigidity of the cable cross section in the presence of axial tension and curvatures

$$\overline{GJ_{ef}} = \overline{GJ} + \frac{\overline{EI_{11}} + \overline{EI_{22}}}{\overline{EA}} T + [(\overline{EI_{11}} + \overline{EI_{22}})\theta_n - \iint E(\theta_1^2 + \theta_2^2)\theta_1 \mathrm{d}\theta_1 \mathrm{d}\theta_2]\kappa \tag{18e}$$

and where $f_i^{(I)*}$ represent the components of inertial forces. We note that the inertial moment components are omitted in Eqs. (17) as the influence of these quantities on local stability is judged to be negligible.

If the position vector $r(s,t)$ of the reference curve $\mathcal{C}$ is expressed in the form

$$r = \hat{r} + uA_1 + vA_2 + wA_3 \tag{19}$$

then the analysis in Ref. 1 implies the following curvature strain displacement relations upon retention of only linear terms in the perturbation quantities,

$$\kappa_1^* = u'' + \kappa^2 u - \kappa w', \quad \kappa_2^* = v'' - \kappa\beta$$
$$\kappa_3^* = \beta' + \kappa v', \quad e^* = w' - \kappa u \tag{20}$$

where β is the rotation of the cross section about the A_3 axis. In addition, the inertial forces and moments are given by

$$f_1^{(I)} = -m\ddot{u}, \quad f_2^{(I)} = -m(\ddot{v} + \theta_m\ddot{\beta})$$
$$f_3^{(I)} = -m(\ddot{w} - \theta_m\ddot{u}), \quad m_3^{(I)} = -m(\theta_m\ddot{v} + \theta_g^2\ddot{\beta}) \tag{21}$$

where $(\ )^{\cdot} = \partial(\ )\partial t$ and where, with $\rho(\theta_1,\theta_2)$ representing the density distribution of the constituent elements on the cable cross section,

$$m = \iint\rho \mathrm{d}\theta_1 \mathrm{d}\theta_2, \quad \theta_m = \iint\rho\theta_1 \mathrm{d}\theta_1 \mathrm{d}\theta_2/m$$
$$\theta_g^2 = \iint\rho(\theta_1^2 + \theta_2^2)\mathrm{d}\theta_1 \mathrm{d}\theta_2/m \tag{22}$$

We have θ_m algebraically positive if the center of mass of the cross section lies in between the shear center and the leading edge.

Next, upon differentiating Eq. (17a) with respect to s, using Eq. (16b) to eliminate $N_2^{*\prime}$, Eq. (13) to express N and M in terms T and κ and Eqs. (18) to write M_1^* and M_3^* in terms of κ_2^* and κ_3^*, and finally Eqs. (20) and (21) to express curvatures and inertial forces in terms of displacements, we obtain

$$\overline{EI_{22}}(v'' - \kappa\beta)'' + [(\overline{EI_{11}} - \overline{EA}\theta_n^2 - \overline{GJ_{ef}})\kappa - T\theta_n](\beta' + \kappa v')' + [(2EI_{11} - 2\overline{EA}\theta_n^2 - \overline{GJ_{ef}})\kappa - 2T\theta_n]'(\beta' + \kappa v') - T(v'' - \kappa\beta) + m(\ddot{v} + \theta_m\ddot{\beta}) - f_2^{(H)*} = 0 \tag{23}$$

In a similar manner, upon combining Eq. (17c) with Eqs. (18) and (21) we find

$$\overline{GJ_{ef}}(\beta' + \kappa v')' + [(\overline{EI_{22}} - \overline{EI_{11}} + \overline{EA}\theta_n^2)\kappa + T\theta_n](v'' - \kappa\beta) - m(\theta_m\ddot{v} + \theta_g^2\ddot{\beta}) + m_3^{(H)*} = 0 \tag{24}$$

Equations (16a) and (16c) can be reduced, in an analogous fashion, into two coupled equations for u and w. However, these in-plane disturbances and their consequences are of less significance as far as the stability of the cable is concerned compared to the out-of-plane disturbances v and β.

In Eqs. (23) and (24) $f_2^{(H)*}$ and $m_3^{(H)*}$ represent the hydrodynamic lift force and the hydrodynamic moment per unit length of the cable. Expressions for $f_2^{(H)*}$ and $m_3^{(H)*}$ are taken in accordance with the velocity component strip theory for hydrofoils oscillating in a simple harmonic fashion. These expressions, as given in Ref. 8, are

$$f_2^{(H)*} = C_{L\beta}\rho_w bV_n C(k)\{V_n(\beta - v'\tan\varphi) - \dot{v} + (1/2 - a)b(\dot{\beta} + V_n\beta'\tan\varphi)\} + C_{NC}\rho_w b^2\{V_n(\dot{\beta} - \dot{v}'\tan\varphi) - \ddot{v} - ab(\ddot{\beta} + V_n\dot{\beta}'\tan\varphi)\} \tag{25}$$

$$m_3^{(H)*} = C_{L\beta}\rho_w b^2 V_n(a + 1/2)C(k)\{V_n(\beta - v'\tan\varphi) - \dot{v} + (1/2 - a)b(\dot{\beta} + V_n\beta'\tan\varphi)\} + C_{NC}\rho_w b^2 \times \{V_n[(a - 1/2)\dot{\beta} - \dot{v}'a\tan\varphi] - a\ddot{v} - 1/2 V_n^2\beta'\tan\varphi) - (1/8 + a^2)b(\ddot{\beta} + V_n\dot{\beta}'\tan\varphi)\} \tag{26}$$

where $a \times b$ represents the distance between the shear center and the midchord with a positive if the midchord lies in between the shear center and the leading edge. In the case of a cable, the lift constants $C_{L\beta}$ and C_{NC} may have values different from their values $C_{L\beta}=2C_{NC}=2\pi$ corresponding to a two-dimensional flat plate. In Eqs. (25) and (26) V_n and ρ_w represent the component of fluid velocity normal to $\mathcal{C}$ and the density of the fluid, respectively. The parameter k is given by

$$k=\omega b/V_n \tag{27}$$

where ω is the circular frequency of the simple harmonic motion. In Eqs. (25) and (26) $C(k)$ is the Theodorsen's function, defined as

$$C(k)=H_1^{(2)}(k)/[H_1^{(2)}(k)+iH_0^{(2)}(k)] \tag{28}$$

where $i=\sqrt{-1}$ and $H_n^{(2)}$ are the Hankel functions.

The system of Eqs. (23-26) are to be supplemented by an appropriate system of boundary conditions. In what follows we consider two cases of boundary conditions. In the first case we assume that both ends of the cable are hinged as far as lateral displacements are concerned and are constrained from rotation about the elastic axis. In the second case we assume that the upper end of the cable satisfies the same conditions as in the first case, but the lower end is attached by means of a ball joint to a passive body. By passive we imply a body for which, under small displacements, the body drag and lift vectors remain constant in magnitude and direction.

The foregoing statements may be expressed as,

Case 1

$$\begin{aligned} &s=0;\quad v=0,\quad \overline{EI_{22}}v''=0,\quad \overline{GJ_{ef}}\beta=0\\ &s=l;\quad v=0,\quad \overline{EI_{22}}v''=0,\quad \overline{GJ_{ef}}\beta=0 \end{aligned} \tag{29}$$

Case 2

$$\begin{aligned} &s=0;\quad v=0,\quad \overline{EI_{22}}v''=0,\quad \overline{GJ_{ef}}\beta=0\\ &s=l;\quad \overline{EI_{22}}v''=0,\quad \overline{EI_{22}}(v''-\kappa\beta)'-Tv'=0\\ &\qquad \overline{GJ_{ef}}(\beta'+\kappa v')=0 \end{aligned} \tag{30}$$

The analysis of stability now centers upon an investigation of the solution of Eqs. (23) and (24) subject to any particular case of boundary conditions listed previously. Substantial simplification of these equations is possible through the extension of the order-of-magnitude analysis based on the smallness of the chord to length ratio, initiated in Sec. II.

IV. Asymptotic Approximations

The smallness of the chord to length ratio of the cable may be used to assess the magnitude of various terms in the stability Eqs. (23) and (24) and to simplify these equations through neglect of unimportant terms. To this end, we redefine ϵ as

$$\epsilon=\pi b/L \tag{31a}$$

where L is an "effective length" of the cable (depending on the boundary conditions). Furthermore, we add to the nondimensional quantities listed in Eqs. (7) the following parameters:

$$\begin{aligned} &\xi=\pi s/L,\ \tilde{v}=v/b,\ \tilde{\beta}=\beta,\ \tilde{\theta}_m=\theta_m/b,\tilde{\theta}_g=\theta_g/b\\ &\tilde{B}_2=\pi\overline{EI_{22}}/\bar{T}bL,\ \tilde{C}=\overline{GJ_{ef}}/\bar{T}b^2,\tilde{V}_n=V_n/\bar{V}_n\\ &\omega_0=(\bar{T}\pi^2/mL^2)^{1/2},\ \tau=\omega_0 t,\ k_0=\omega_0 b/\bar{V}_n,\ \alpha=C_{NC}/C_{L\beta}\\ &\tilde{F}=C_{L\beta}\rho_w\bar{V}_n^2L^2/\pi^2\bar{T},\gamma=L^2/\pi^2Rb,\ \delta=L/\pi R \end{aligned} \tag{31b}$$

where $\bar{V}_n$ is an average value of V_n. The nondimensional variables ζ introduced in Eqs. (7) and ξ are such that

$$\partial(\tilde{v},\tilde{\beta})/\partial\xi=O(1),\ \mathrm{d}(\tilde{T},\tilde{\kappa})/\mathrm{d}\zeta=O(1) \tag{32}$$

In terms of the nondimensional quantities in Eqs. (31) and (7), the differential Eqs. (23) and (24) and the boundary conditions of Eqs. (29) and (30) become

$$\begin{aligned} &\epsilon\tilde{B}_2(\tilde{v},_{\xi\xi}-\gamma\tilde{\kappa}\tilde{\beta}),_{\xi\xi}+[\delta(\tilde{B}_1-\epsilon\tilde{C})\tilde{\kappa}-\tilde{\theta}_n\tilde{T}](\tilde{\beta},_\xi+\epsilon\delta\tilde{\kappa}\tilde{v},_\xi),_\xi\\ &-\tilde{T}(\tilde{v},_{\xi\xi}-\gamma\tilde{\kappa}\tilde{\beta})+[\delta^2(2\tilde{B}_1-\epsilon\tilde{C})\tilde{\kappa},_\xi-2\delta\tilde{\theta}_n\tilde{T},_\tau](\tilde{\beta},_\xi+\epsilon\delta\tilde{\kappa}\tilde{v},_\xi)\\ &+\tilde{v},_{\tau\tau}+\tilde{\theta}_m\tilde{\beta},_{\tau\tau}-\tilde{F}\tilde{V}_nC(k)[\tilde{V}_n(\tilde{\beta}-\epsilon\tilde{v},_\xi\tan\varphi)-k_0\tilde{v},_\tau\\ &+(1/2-a)(k_0\tilde{\beta},_\tau+\epsilon\tilde{V}_n\tilde{\beta},_\xi\tan\varphi)]-\tilde{F}\alpha[\tilde{V}_nk_0(\tilde{\beta},_\tau-\epsilon\tilde{v},_{\tau\xi}\tan\varphi)\\ &-k_0^2v,_{\tau\tau}-a(k_0^2\beta,_{\tau\tau}+\epsilon k_0V_n\beta,_{\tau\xi}\tan\varphi)]=0 \end{aligned} \tag{33}$$

$$\begin{aligned} &\tilde{C}(\tilde{\beta},_\xi+\epsilon\delta\tilde{\kappa}\tilde{v},_\xi),_\xi+[\delta(\tilde{B}_2-\tilde{B}_1)\tilde{\kappa}+\tilde{\theta}_n\tilde{T}](\tilde{v},_{\xi\xi}-\gamma\tilde{\kappa}\tilde{\beta})\\ &\quad-\tilde{\theta}_m\tilde{v},_{\tau\tau}-\theta_g^2\tilde{\beta},_{\tau\tau}+\tilde{F}\tilde{V}_nC(k)(1/2+a)[\tilde{V}_n(\tilde{\beta}-\epsilon\tilde{v},_\xi\tan\varphi)\\ &\quad-k_0\tilde{v},_\tau+(1/2-a)(k_0\tilde{\beta},_\tau+\epsilon\tilde{V}_n\tilde{\beta},_\xi\tan\varphi)]-F\alpha\{\tilde{V}_nk_0\\ &\quad\times[(1/2-a)\tilde{\beta},_\tau+\epsilon a\tilde{v},_{\tau\xi}\tan\varphi]+ak_0^2\tilde{v},_{\tau\tau}+1/2\epsilon\tilde{V}_n^2\tilde{\beta},_\xi\tan\varphi\\ &\quad+(1/8+a^2)(k_0^2\tilde{\beta},_{\tau\tau}+\epsilon\tilde{V}_nk_0\tilde{\beta},_{\tau\xi}\tan\varphi)\}=0 \end{aligned} \tag{34}$$

Case 1

$$\tilde{v}=0,\ \epsilon\tilde{B}_2\tilde{v},_{\xi\xi}=0,\ \tilde{C}\tilde{\beta}=0\ \text{at}\ \xi=0,\ \xi=\pi l/L \tag{35}$$

Case 2

$$\begin{aligned} &\tilde{v}=0,\ \epsilon\tilde{B}_2\tilde{v},_{\xi\xi}=0,\ \tilde{C}\tilde{\beta}=0\ \text{at}\ \xi=0,\ \epsilon B_2\tilde{v},_{\xi\xi}=0,\\ &\epsilon\tilde{B}_2(\tilde{v},_{\xi\xi}-\gamma\tilde{\kappa}\tilde{\beta}),_\xi-\tilde{T}\tilde{v},_\xi=0,\tilde{C}(\tilde{\beta}+\epsilon\delta\tilde{\kappa}\tilde{v},_\xi),_\xi=0\ \text{at}\ \xi=\pi l/L \end{aligned} \tag{36}$$

From Eqs. (33) and (34) we note that when the dynamic state of the cable is such that $\tilde{v}=O(\tilde{\beta})$, the parameter $\gamma=L^2/\pi^2Rb$ must be of the order unity. Furthermore, we have $\tilde{F}=O(1)$. Corresponding to this situation, we have from Eqs. (31)

$$\beta/v=O(1),\ \delta\equiv L/\pi R=O(\epsilon),\ C_{L\beta}\rho_w\bar{V}_n^2bL/\pi\bar{T}=O(\epsilon) \tag{37}$$

The third relation in the preceding shows that the total hydrodynamic force on the cable is small compared to the cable tension. This will be true in general when the purpose of the cable is to tow a non-self-propellant body underwater. As the drag force responsible for the curvature of the cable is also small (compared to the cable tension) in this event, the second relation in Eq. (37) is consistent with the remaining relations in that equation. In what follows a cable system satisfying Eq. (37) will be referred to as a towed system.

On the other hand, when the body is self-propellent the tension in the cable is mainly due to the drag force on the cable and the cable curvature will be large. In this case the hydrodynamic forces may be signficant in comparison to the cable tension and the order-of-magnitude relations of Eq. (37) is to be replaced by

$$\tilde{\beta}/\tilde{v}=O(\epsilon),\ \delta\equiv L/\pi R=O(1),\ C_{L\beta}\rho_w\bar{V}_n^2bL/\pi\bar{T}=O(1) \tag{38}$$

In the subsequent analysis a cable system satisfying Eq. (38) will be referred to as a tethered system.

V. Stability of a Towed Cable System

In the light of the order-of-magnitude relations of Eq. (37) we consider asymptotic expansions of $\tilde{v}$ and $\tilde{\beta}$ in terms of the parameter ϵ. Retaining only the zeroth-order terms in Eqs. (33-36) the equations of stability may be written as,

$$[m+C_{NC}\rho_w b^2]v,_{tt}+\rho_w bV_n C_{L\beta}v,_t-Tv,_{ss}+[m\theta_m+C_{NC}\rho_w b^3 a]\beta,_{tt}+\rho_w b^2 V_n[(a-\tfrac{1}{2})C_{L\beta}-C_{NC}]\beta,_t-T\theta_n\beta,_{ss}+[T\kappa-\rho_w bV_n^2 C_{L\beta}]\beta=0 \tag{39}$$

$$[m\theta_m+C_{NC}\rho_w b^3 a]v,_{tt}+\rho_w b^2 V_n(a+\tfrac{1}{2})C_{L\beta}v,_t-[(B_2+C-B_1)\kappa+T\theta_n]v,_{ss}+[m\theta_g^2+C_{NC}\rho_w b^4(\tfrac{1}{8}+a^2)]\beta,_{tt}+\rho_w b^3 V_n(a-\tfrac{1}{2})[(a+\tfrac{1}{2})C_{L\beta}-C_{NC}]\beta,_t-C\beta,_{ss}+[T\theta_n\kappa-\rho_w b^2 C_{L\beta}V_n^2(\tfrac{1}{2}+a)]\beta=0 \tag{40}$$

where

$$B_2=\overline{EI_{22}},\quad B_1=\overline{EI_{11}}-\overline{EA}\theta_n^2,\quad C=\overline{GJ_{ef}} \tag{41}$$

We note that in Eq. (40) we have retained small-order terms in the coefficient of $v,_{ss}$, as these terms containing the structural properties B_i and C play an important role in the stability characteristics of the system. Furthermore, we have set $C(k)\approx 1$ considering the low frequencies of oscillations encountered in long cable systems.

The associated boundary conditions are

Case 1

$$v=0,\ \beta=0 \text{ at } s=0 \text{ and at } s=l$$

Case 2

$$v=0,\ \beta=0 \text{ at } s=0;\quad v'=0,\ \beta'=0 \text{ at } s=l \tag{42}$$

Solutions of Eqs. (39) and (40) are obtained by means of Galerkin approximation. We assume the fundamental mode of instability is of the form

$$v=v_0 e^{\lambda t}\sin\pi s/L,\quad \beta=\beta_0 e^{\lambda t}\sin\pi s/L \tag{43}$$

where λ is a complex constant and where, in order to satisfy the boundary condition, L must be taken as

Case 1

$$L=l \tag{44}$$

Case 2

$$L=2l \tag{45}$$

Let $\mathcal{E}_v$ and $\mathcal{E}_\beta$ denote the pointwise error in satisfying the Eqs. (39) and (40), respectively. Then, we may write,

$$\mathcal{E}_v=(\mathcal{E}_{vv}v_0+\mathcal{E}_{v\beta}\beta_0)e^{\lambda t}\sin\pi s/L \tag{46}$$

$$\mathcal{E}_\beta=(\mathcal{E}_{\beta v}v_0+\mathcal{E}_{\beta\beta}\beta_0)e^{\lambda t}\sin\pi s/L \tag{47}$$

where

$$\mathcal{E}_{vv}=(m+C_{NC}\rho_w b^2)\lambda^2+\rho_w bV_n C_{L\beta}\lambda+\pi^2 T/L^2$$

$$\mathcal{E}_{v\beta}=(m\theta_m+C_{NC}\rho_w b^3 a)\lambda^2+\rho_w b^2 V_n[(a-\tfrac{1}{2})C_{L\beta}-C_{NC}]\lambda+\pi^2 T\theta_n/L^2+T\kappa-\rho_w bV_n^2 C_{L\beta}$$

$$\mathcal{E}_{\beta v}=(m\theta_m+C_{NC}\rho_w b^3 a)\lambda^2+\rho_w b^2 V_n(a+\tfrac{1}{2})C_{L\beta}\lambda+\pi^2[(B_2+C-B_1)\kappa+T\theta_n]/L^2$$

$$\mathcal{E}_{\beta\beta}=[m\theta_g^2+C_{NC}\rho_w b^4(\tfrac{1}{8}+a^2)]\lambda^2+\rho_w b^3 V_n(a-\tfrac{1}{2})[(a+\tfrac{1}{2})C_{L\beta}-C_{NC}]\lambda+\pi^2 C/L^2+T\theta_n\kappa-\rho_w b^2 V_n^2(\tfrac{1}{2}+a)C_{L\beta} \tag{48}$$

Next, by setting

$$\int_0^L \mathcal{E}_v\sin\pi\frac{s}{L}=0,\quad \int_0^L \mathcal{E}_\beta\sin\pi\frac{s}{L}=0 \tag{49}$$

we obtain a set of homogeneous simultaneous equations for v_0 and β_0,

$$\{[M]\lambda^2+[D]\lambda+[K]\}\begin{Bmatrix}v_0\\ \beta_0\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{50}$$

where M, D, and K are 2×2 matrices with elements,

$$M_{11}=m+C_{NC}\rho_w b^2$$
$$M_{12}=m\theta_m+C_{NC}\rho_w b^3 a$$
$$M_{21}=M_{12}$$
$$M_{22}=m\theta_g^2+C_{NC}\rho_w b^4(\tfrac{1}{8}+a^2)$$
$$D_{11}=C_{L\beta}\rho_w b\bar{V}_n$$
$$D_{12}=[(a-\tfrac{1}{2})C_{L\beta}-C_{NC}]\rho_w b^2\bar{V}_n$$
$$D_{21}=C_{L\beta}\rho_w b^2(a+\tfrac{1}{2})\bar{V}_n$$
$$D_{22}=(a-\tfrac{1}{2})[(a+\tfrac{1}{2})C_{L\beta}-C_{NC}]\rho_w b^3\bar{V}_n$$
$$K_{11}=\pi^2\bar{T}/L^2$$
$$K_{12}=\pi^2\bar{T}\theta_n/L^2+\bar{T}/R-\rho_w b\bar{V}_n^2 C_{L\beta}$$
$$K_{21}=\pi^2[(B_2+\bar{C}-B_1)/R+\bar{T}\theta_n]L^2$$
$$K_{22}=\pi^2\bar{C}/L^2+\bar{T}\theta_n/R-\rho_w b^2\bar{V}_n^2(\tfrac{1}{2}+a)C_{L\beta} \tag{51}$$

and where,

$$\left(\bar{T},\frac{1}{R},\bar{C},\bar{V}_n^2\right)=\frac{2}{L}\int_0^L(T,\kappa,C,V_n^2)\sin^2\frac{\pi s}{L} \tag{52}$$

Here, based on the assumption of slowly varying steady-state variables we have taken

$$\frac{2}{L}\int_0^L(T\kappa,\kappa^2,V_n)\sin^2\frac{\pi s}{L}ds\approx\frac{\bar{T}}{R},\frac{1}{R^2},\bar{V}_n \tag{53}$$

The characteristic equation for λ follows upon setting the determinant of the matrix $[M]\lambda^2+[D]\lambda+[K]$ equal to zero. We write the characteristic equation in the form,

$$\sum_{n=0}^{4}C_n\lambda^n=0 \tag{54}$$

where

$$C_0=K_{11}K_{22}-K_{12}K_{21}$$
$$C_1=D_{11}K_{22}+D_{22}K_{11}-D_{12}K_{21}-D_{21}K_{12}$$

$$C_2 = M_{11}K_{22} + M_{22}K_{11} + D_{11}D_{22} - D_{21}D_{12} - M_{12}K_{21} - M_{21}K_{12}$$

$$C_3 = M_{11}D_{22} + M_{22}D_{11} - M_{12}D_{21} - M_{21}D_{12}$$

$$C_4 = M_{11}M_{22} - M_{12}M_{21} \quad (55)$$

Depending on the solutions of Eq. (54) we have, when

$Re\lambda \leq 0$: no flutter or divergence (56)

$Re\lambda > 0$, $Im\lambda = 0$: divergence (57)

$Re\lambda > 0$, $Im\lambda \neq 0$: flutter (58)

Thus, $Re\lambda \leq 0$ is a sufficient condition for stability. According to Routh-Hurwitz criteria, $Re\lambda \leq 0$ if the coefficients C_n satisfy the following conditions

$$C_n > 0 \ (<0), \ n = 0, 1, \ldots 4 \quad (59)$$

$$C_1C_2C_3 - C_0C_3^2 - C_4C_1^2 > 0 \ (<0) \quad (60)$$

If the inequalities (59) and (60) are satisfied, the system is stable; on the other hand, if they are not satisfied either flutter or divergence may occur. In the inequalities (57) and (58) we distinguish the two critical states; $Re\lambda = 0$, $Im\lambda = 0$, corresponding to the threshold between static stability and divergence and $Re\lambda = 0$, $Im\lambda \equiv \omega \neq 0$, corresponding to the threshold between bounded and unbounded oscillations (flutter). Next, we examine these two states separately.

VI. Divergence and Flutter

For a nontrivial solution of Eq. (54) when $\lambda = 0$ we must have

$$C_0 = K_{11}K_{22} - K_{12}K_{21} = 0 \quad (61)$$

Use of Eqs. (51) in Eq. (61) gives

$$C_{L\beta}\rho_w b^2 \bar{V}_n^2 \left[\frac{1}{2} + a - \frac{\theta_n}{b} - \frac{B_2 + \bar{C} - B_1}{\bar{T}Rb}\right] = \frac{\pi^2}{L^2}(\bar{C} - \bar{T}\theta_n^2) - \frac{\pi}{L}\frac{B_2 + \bar{C} - B_1}{R}\left(\frac{\pi\theta_n}{L} + \frac{L}{\pi R}\right) \quad (62)$$

In Eq. (62), generally $\bar{C} \gg \bar{T}\theta_n^2$, and in accordance with our earlier assumptions, $\pi\theta_n/L$, $L/\pi R \approx O(\epsilon)$. Then, Eq. (62) reduces to

$$C_{L\beta}\rho_w b^2 \bar{V}_n^2 = \frac{\pi^2 \bar{C}/L^2}{\frac{1}{2} + a - \theta_n/b - (\overline{EI}_{22} + \bar{C} - \overline{EI}_{11} + \overline{EA}\theta_n^2)/\bar{T}Rb} \quad (63)$$

which defines critical divergence velocity in an implicit form as $\bar{T}$ and R are functions of $\bar{V}_n$. However, Eq. (63) provides a sufficient condition for freedom from divergence, in the form

$$\bar{T}Rb\left[\frac{1}{2} + a - \frac{\theta_n}{b}\right] \leq \overline{EI}_{22} + \bar{C} - \overline{EI}_{11} + \overline{EA}\theta_n^2 \quad (64)$$

From Eq. (63) there follow as optimum conditions for stability with respect to divergence, a) $\theta_n - (\frac{1}{2} + a)b$ is as large as possible, b) $\overline{EI}_{22} > \overline{EI}_{11} - \overline{EA}\theta_n^2$, and c) $\bar{C}$ is as large as possible.

Effects of Sweep and Chord to Length Ratio on Divergence Velocity

The asymptotic expansions in Eqs. (39) and (40) do not reveal how small $\epsilon = \pi b/L$ must be for a given error in the divergence velocity. An estimate for the influence of ϵ on the divergence velocity may be obtained as follows. We consider a hypothetical cable for which

$$\overline{EI}_{22} = \overline{EI}_{11} - \overline{EA}\theta_n^2 = \overline{GJ}_{ef}$$

$$R = \infty, \ \varphi = \text{constant} = \bar{\varphi}$$

$$\theta_n = 0, \ a = \frac{1}{2}, \ T = \text{constant} = \bar{T} \quad (65)$$

Then, Eqs. (23-26) reduce to

$$\bar{T}\frac{d^2v}{ds^2} + C_{L\beta}\rho_w b V_n^2\left[\beta - \frac{dv}{ds}\tan\bar{\varphi}\right] = 0 \quad (66)$$

$$\overline{GJ}_{ef}\frac{d^2\beta}{ds^2} + C_{L\beta}\rho_w b^2 V_n^2\left[\beta - \frac{dv}{ds}\tan\bar{\varphi}\right] = 0 \quad (67)$$

We seek nontrivial solutions of Eqs. (66) and (67) subject to the boundary conditions of case 1, that is,

$$v(0) = v(l); \quad \beta(0) = \beta(l) = 0 \quad (68)$$

Eliminating the $C_{L\beta}$ term we deduce from Eqs. (66) and (67)

$$\frac{d^2}{ds^2}[\bar{T}bv - \overline{GJ}_{ef}\beta] = 0 \quad (69)$$

Integration of Eq. (69) twice and use of the boundary conditions of Eq. (68) gives

$$\overline{GJ}_{ef}\beta = \bar{T}bv \quad (70)$$

From Eqs. (70) and (66) we have

$$\frac{d^2v}{ds^2} - \frac{C_{L\beta}}{\bar{T}}\rho_w b V_n^2\left[\tan\bar{\varphi}\frac{dv}{ds} - \frac{\bar{T}b}{\overline{GJ}_{ef}}v\right] = 0 \quad (71)$$

Now, let

$$v = e^{ps}\bar{v}(s), \quad p = C_{L\beta}\rho_w b V_n^2 \tan\bar{\varphi}/2\bar{T} \quad (72)$$

then $\bar{v}$ satisfies the differential equation

$$\frac{d^2\bar{v}}{ds^2} + \frac{C_{L\beta}\rho_w b^2 V_n^2}{\overline{GJ}_{ef}}\left[1 - \frac{\overline{GJ}_{ef}C_{L\beta}\rho_w V_n^2 \tan\bar{\varphi}}{4\bar{T}^2}\right]\bar{v} = 0 \quad (73)$$

The desired solution of Eq. (73) satisfying the boundary conditions of Eq. (68) is $\bar{v} = A\sin\pi s/l$. A nontrivial solution exists if and only if

$$C_{L\beta}\rho_w b^2 V_n^2\left[1 - \frac{\overline{GJ}_{ef}C_{L\beta}\rho_w V_n^2 \tan^2\bar{\varphi}}{4\bar{T}^2}\right] = \frac{\pi^2\overline{GJ}_{ef}}{l^2} \quad (74)$$

From Eq. (63) we have, corresponding to the assumptions in Eqs. (65), the critical divergence velocity given by

$$C_{L\beta}\rho_w b^2 V_{no}^2 = \pi^2\overline{GJ}_{ef}/l^2 \quad (75)$$

where the subscript o denotes the condition $\epsilon = o$. From Eqs. (74) and (75), the critical velocity $V_{n\epsilon}$, when $\epsilon \neq o$, may be written as

$$\left(\frac{V_{n\epsilon}}{V_{no}}\right)^2 = 1 + \eta\left(\frac{V_{n\epsilon}}{V_{no}}\right)^2; \quad \eta = \left(\frac{\overline{GJ}_{ef}}{2\bar{T}b^2}\tan\bar{\varphi}\right)^2\epsilon^2 \quad (76)$$

Upon solving the quadratic Eq. (76) for $(V_{n\epsilon}/V_{no})^2$ and expanding the result in powers of ϵ we find

$$\frac{V_{n\epsilon}}{V_{no}} = 1 + \left(\frac{\overline{GJ}_{ef}}{\bar{T}b^2}\right)^2\frac{\tan^2\bar{\varphi}}{8}\epsilon^2 + O(\epsilon^4) \quad (77)$$

From Eq. (77) we estimate that the error in V_{no} is less than 1% if

$$\frac{\pi b}{l} \equiv \epsilon < \frac{\sqrt{8}}{10} \frac{\bar{T} b^2}{G J_{ef}} \cot\bar{\varphi} \tag{78}$$

Furthermore, Eq. (77) shows that V_{no} is a conservative approximation to $V_{n\epsilon}$ as $V_{n\epsilon} > V_{no}$.

Flutter

When $Re\lambda = 0$, $Im\lambda \equiv \omega \neq 0$, we write Eq. (50) in the form

$$\left\{1+\mu\left[1-\left(\frac{\omega_v}{\omega_\beta}\right)^2\left(\frac{\omega_\beta}{\omega}\right)^2\right]-\frac{i}{\alpha k_\beta}\frac{\omega_\beta}{\omega}\right\}v_0 + \left\{a+\mu\left[\tilde{\theta}_m-\left(\frac{\omega_{v\beta}}{\omega_\beta}\right)^2\left(\frac{\omega_\beta}{\omega}\right)^2\right]-\frac{i}{\alpha k_\beta}\left(a-\frac{1}{2}-\alpha\right)\left(\frac{\omega_\beta}{\omega}\right) + \frac{1}{\alpha k_\beta^2}\left(\frac{\omega_\beta}{\omega}\right)^2\right\}\beta_0=0 \tag{79}$$

$$\left\{a+\mu\left[\tilde{\theta}_m-\left(\frac{\omega_{\beta v}}{\omega}\right)^2\left(\frac{\omega_\beta}{\omega}\right)^2\right]-\frac{i}{\alpha k_\beta}\left(a+\frac{1}{2}\right)\left(\frac{\omega_\beta}{\omega}\right)\right\}v_0 + \left\{\frac{1}{8}+a^2+\mu\tilde{\theta}_g^2\left[1-\left(\frac{\omega_\beta}{\omega}\right)^2\right] - \frac{i}{\alpha k_\beta}\left(a+\frac{1}{2}-\alpha\right)\left(a-\frac{1}{2}\right)\frac{\omega_\beta}{\omega}+\frac{a+½}{\alpha k_\beta^2}\left(\frac{\omega_\beta}{\omega}\right)^2\right\}\beta_0=0 \tag{80}$$

where

$$\mu=\frac{m}{C_{NC}\rho_w b^2}\ \text{(the mass ratio)},\quad k_\beta=\frac{\omega_\beta b}{\bar{V}_n}$$

$$\omega_v^2=\frac{\pi^2\bar{T}}{mL^2},\quad \omega_\beta^2=\frac{\pi^2\bar{C}}{mL^2\theta_g^2}\left[1+\frac{\bar{T}b^2}{C}\frac{L^2}{\pi^2 Rb}\right]$$

$$\omega_{v\beta}^2=\frac{\pi^2\bar{T}}{mL^2}\left[\tilde{\theta}_n+\frac{L^2}{\pi^2 Rb}\right],\quad \omega_{\beta v}^2=\frac{\pi^2\bar{T}}{mL^2}\left[\tilde{\theta}_n+\frac{B_2-B_1+C}{\bar{T}Rb}\right] \tag{81}$$

In Eqs. (81) ω_v and ω_β represent, respectively, the frequencies associated with the uncoupled lateral and torsional oscillations of the cable. The quantities $\omega_{v\beta}$ and $\omega_{\beta v}$ represent the coupling effects due to the cable tension and curvature.

Setting the determinant of the coefficients of v_0 and β_0 to zero we may obtain a real and imaginary equation for the quantity (ω_β/ω). The imaginary equation yields,

$$\left(\frac{\omega_\beta}{\omega}\right)^2=\frac{\tilde{\theta}_g^2+(a-½)(a-\alpha+½)-\tilde{\theta}_m(2a-\alpha)+(\alpha-¼)/2\mu}{\tilde{\theta}_g^2+[(a-½)(a-\alpha+½)\omega_v^2-(a-\alpha-½)\omega_{\beta v}^2-(a+½)\omega_{v\beta}^2]/\omega_\beta^2} \tag{82}$$

From the defining relations of Eqs. (81) we note that $\omega_{v\beta}^2$ and $\omega_{\beta v}^2$ are small compared to ω_v^2. Thus, Eq. (82) may be simplified to

$$\left(\frac{\omega_\beta}{\omega}\right)^2=\frac{\tilde{\theta}_g^2+(a-½)(a-\alpha+½)-\tilde{\theta}_m(2a-\alpha)+(\alpha-¼)/2\mu}{\tilde{\theta}_g^2+(a-½)(a-\alpha+½)\omega_v^2/\omega_\beta^2} \tag{83}$$

The real equation can similarly be solved in terms of ω_β/ω and k_β. We now seek, as in aeroelasticity, a critical mass ratio μ_{cr} below which the system is flutter free. If μ_{cr} exists, then $\bar{V}_n=\infty$, $\mu=\mu_{cr}$. Let us, for convenience, assume that $\bar{T}$ and R are independent of $\bar{V}_n$. Then ω_v and ω_β are independent of $\bar{V}_n$ and Eq. (83) implies ω is independent of $\bar{V}_n$. Thus, $k_\beta \to 0$ as $\bar{V}_n \to \infty$. The solution of the real equation for this case is,

$$\left(\frac{\omega_\beta}{m}\right)^2=\frac{a+½-\tilde{\theta}_m-(2\mu_{cr})^{-1}}{(a+½)(\omega_v^2-\omega_{\beta v}^2)/\omega_\beta^2}\approx\frac{a+½-\tilde{\theta}_m-(2\mu_{cr})^{-1}}{(a+½)\omega_v^2/\omega_\beta^2} \tag{84}$$

From Eqs. (84) and (83) we solve for μ_{cr} to obtain

$$\mu_{cr}=\frac{½+½(\alpha+a^2-a/4-⅜)(\omega_v/\omega_\beta\tilde{\theta}_g)^2}{½+a-\tilde{\theta}_m-[(½+a)\tilde{\theta}_g^2+(\alpha-a^2-a-¼)\tilde{\theta}_m](\omega_v/\omega_\beta\tilde{\theta}_g)^2} \tag{85}$$

In the case of flat plates supported elastically in translation and rotation we have $C_{NC}=\pi$ and $C_{L\beta}=2\pi$. Then $\alpha=½$ and the expression in Eq. (85) reduces to the result obtained in Ref. 9. In general, as a sufficient condition for freedom from flutter we have $\mu<\mu_{cr}$.

Equation (85) shows that μ_{cr} can be made as large as possible by varying the location of the center of mass $\tilde{\theta}_m$. The asymptotic value of $\tilde{\theta}_m$ when $\mu_{cr}\to\infty$ is given by

$$\tilde{\theta}_{m\,cr}=\frac{(½+a)(1-\omega_v^2/\omega_\beta^2)}{1-(a^2+a+¼-\alpha)(\omega_v/\omega_\beta\tilde{\theta}_g)^2} \tag{86}$$

From Eqs. (86) and (85) we conclude that the cable is free from flutter when $\tilde{\theta}_m>\tilde{\theta}_{m\,cr}$.

VII. Stability of Tethered Cable Systems

When the cable tension is of the order of the total hydrodynamic force on the cable, we have, from Eqs. (33, 34, and 38) as the equations of stability,

$$v,_{\tau\tau}-\bar{T}(\bar{v},_{\xi\xi}-\delta\bar{\kappa}\tilde{\beta})-\bar{F}\bar{V}_n^2(\tilde{\beta}-\bar{v},_\xi\tan\varphi)+k_0\bar{F}\bar{V}_n v,_\tau=0 \tag{87}$$

$$\tilde{\theta}_m\bar{v},_{\tau\tau}-[\bar{T}\tilde{\theta}_n+\delta(\bar{B}_2-\bar{B}_1)\bar{\kappa}](\bar{v},_{\xi\xi}-\delta\bar{\kappa}\tilde{\beta})-\epsilon\bar{C}\tilde{\beta},_{\xi\xi} -\bar{F}\bar{V}_n^2(½+a)(\tilde{\beta}-\bar{v},_\xi\tan\varphi)+\bar{F}\bar{V}_n(½+a)k_0\bar{v},_\tau=0 \tag{88}$$

where we use the nondimensional quantities introduced in Eqs. (31), except for the following:

$$\tilde{\beta}=\beta L/\pi b,\ \bar{F}=C_{L\beta}\rho_w\bar{V}_n^2 bL/\pi\bar{T},\ k_0=\omega_0 L/\pi\bar{V}_n \tag{89}$$

which are, now, of order unity. We have retained a "small"-order term containing the elastic parameter $\bar{C}\approx GJ/\bar{T}b^2$ to assess its effect on the stability of the cable.

The associated boundary conditions are

Case 1

$$v=\beta=0 \text{ at } \xi=0 \text{ and at } \xi=\pi \tag{90}$$

Case 2

$$v=\beta=0 \text{ at } \xi=0;\ v,_\xi=\beta,_\xi=0 \text{ at } \xi=\pi/2 \tag{91}$$

Once again we use Galerkin approximation with modes of instability of the form,

$$\tilde{v}=v_0 e^{\lambda\tau}\sin\xi, \quad \tilde{\beta}=\beta_0 e^{\lambda\tau}\sin\xi \tag{92}$$

where the domains of $\tilde{v}$, $\tilde{w}$ are $0\leq\xi\leq\pi$ in case 1 and $0\leq\xi\leq\pi/2$ in case 2.

Substitution of Eq. (92) into Eqs. (87) and (88) and orthogonalization of the errors with respect to sinξ give

$$\{[M]\lambda^2+[D]\lambda+[K]\}\begin{Bmatrix} v_0 \\ \beta_0 \end{Bmatrix}=0 \tag{93}$$

where

$$M_{11}=1,\ M_{12}=0,\ M_{21}=\tilde{\theta}_m,\ M_{22}=0$$

$$D_{11}=k_0\tilde{F}n,\ D_{12}=0,\ D_{21}=(½+a)k_0\tilde{F}n,\ D_{22}=0$$

$$K_{11}=1+\tilde{F}t_1,\ K_{12}=t_2-\tilde{F}$$

$$K_{21}=\tilde{\theta}_n+\delta(\tilde{B}_2-\tilde{B}_1+\epsilon\tilde{C})+(½+a)\tilde{F}t_1$$

$$K_{22}=t_2\tilde{\theta}_n+\delta^2 t_3(\tilde{B}_2-\tilde{B}_1)+\epsilon\tilde{C}-(½+a)\tilde{F} \tag{94}$$

In Eqs. (94) in view of the defining relations of Eqs. (31) we have chosen $\bar{T}$, $\bar{\kappa}$, and $\bar{V}_n$ such that

$$\frac{2L}{\pi l}\int_0^{\pi l/L}(\bar{T},\bar{\kappa},\bar{V}_n^2)\sin^2\xi d\xi=1 \tag{95}$$

In addition, we have used the notation

$$(n,t_1,t_2,t_3)=\frac{2L}{\pi l}\int_0^{\pi l/L}(V_n,V_n^2\cot\xi\tan\varphi,T\kappa,\kappa^2)\sin^2\xi d\xi \tag{96}$$

The characteristic equation corresponding to Eq. (94) comes out to be,

$$C_2\lambda^2+C_1\lambda+C_0=0 \tag{97}$$

where

$$C_2=\tilde{F}(\tilde{\theta}_m-½-a)+t_2(\tilde{\theta}_n-\tilde{\theta}_m)+\delta^2 t_3(\tilde{B}_2-\tilde{B}_1)+\epsilon\tilde{C}$$

$$C_1=k_0 n\tilde{F}[t_2(\tilde{\theta}_n-½-a)+\delta^2 t_3(\tilde{B}_2-\tilde{B}_1)+\epsilon\tilde{C}]$$

$$C_0=\tilde{F}[(1+t_1t_2)(\tilde{\theta}_n-½-a)+\delta(1+\delta t_1t_3)(\tilde{B}_2-\tilde{B}_1)$$
$$+\epsilon t_1\tilde{C}]+[\delta(\delta t_3-t_2)(\tilde{B}_2-\tilde{B}_1)+\epsilon\tilde{C}] \tag{98}$$

From Eq. (97) we have as conditions for stability

$$C_0>0, \quad C_1>0, \quad C_2>0 \tag{99}$$

Equations (98) in conjunction with Eq. (99) gives as sufficient conditions for freedom from instability,

$$\tilde{\theta}_n>\tilde{\theta}_m>½+a, \quad \tilde{B}_2>\tilde{B}_1 \tag{100}$$

where we have used the fact that

$$t_2,\ t_3>0 \tag{101}$$

We note that for accurate evaluation of critical flow speed, in the case of tethered cables a more specific determination of steady state variables is necessary.

VIII. Concluding Remarks

In the foregoing sections the equations governing the equilibrium state as well as the perturbed state of transversely isotropic underwater cables are obtained on the basis of the nonlinear dynamical theory presented in Ref. 1 for the treatment of heterogeneous, elastic rods. The hydrodynamic forces affecting the perturbed motion of the cable are taken in accordance with the velocity component strip theory.[8] Sufficient conditions for freedom from divergence and flutter are obtained in terms of the locations of the center of tension, center of gravity, and the aerodynamic center, and the radius of gyration of the cable cross section. In addition to the torsional rigidity of the cable the out-of-plane as well as the in-plane bending rigidities, cable tension, and the in-plane curvature of the cable axis affect the critical flow speed. We must mention the fact that the behavior of the towed body may have a significant effect on the stability of the cable-body system, although, our boundary conditions are based on the assumption of a passive body, However, the equations of stability presented here may be employed in conjunction with a mathematical model of the towed body in order to analyze the dynamic behavior of a cable-body system.

References

[1] Hegemier, G. and Nair, S., "A Nonlinear Dynamical Theory for Heterogeneous, Anisotropic, Elastic Rods," *AIAA Journal,* Vol. 15, Jan. 1976, pp. 8-15.

[2] Calkins, D.E., "Faired Towline Hydrodynamics," *Journal of Hydronautics,* July 1970, pp. 113-119.

[3] Caserella, M.J. and Parsons, M., "Cable Systems Under Hydrodynamic Loading," *MTS Journal,* Vol. 4, July-Aug. 1970, pp. 27-44.

[4] Cannon, T.C. and Genin, J., "Three-Dimensional Dynamic Behaviour of a Flexible Towed Cable," *Aeronautical Quarterly,* Vol. 23, Aug. 1972, pp. 201-210.

[5] Anderson, G.F., "Tow Cable Loading Functions," *AIAA Journal,* Vol. 5, Feb. 1967, pp. 346-348.

[6] Eames, M.C., "Steady-State Theory of Towing Cables," *Transactions of the Royal Institute of Naval Architecture,* Vol. 110, 1967.

[7] Anderson, G.F., "Optimum Configuration of a Tethering Cable," *Journal of Aircraft,* Vol. 4, March 1967, pp. 261-263.

[8] Bisplinghoff, R.L., Ashley, H., and Halfman, R.L., *Aeroelasticity,* Addison-Wesley, Cambridge, Mass., 1957.

[9] Heller, S.R., "Hydroelasticity," *Advances in Hydroscience,* edited by V.T. Chow, Academic Press, New York, 1964.